# VII.

# ON THE PROPERTIES OF BATTERIES FORMED OF CELLS JOINED UP IN MULTIPLE ARC.

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If three galvanic cells the electromotive forces of which are $e_1$, $e_2$, and $e_3$ respectively, and the internal resistances, $b_1$, $b_2$, and $b_3$, be joined up parallel to each other, the battery thus formed is equivalent — so far as its ability to send electricity through an external circuit is concerned — to a single cell of electromotive force $e_0$, and of internal resistance $b_0$, where

$$e_0 = \frac{e_1 b_2 b_3 + e_2 b_1 b_3 + e_3 b_1 b_2}{b_2 b_3 + b_3 b_1 + b_1 b_2}, \quad b_0 = \frac{b_1 b_2 b_3}{b_2 b_3 + b_1 b_3 + b_1 b_2}.$$

A similar statement * may be made about a battery of any number of cells connected in multiple arc. It is evident, however, that the efficiency † of such a battery of unlike cells is less than that of a single cell which would do the same amount of useful work under the same external circumstances.

I have had occasion of late to consider some relations between the strengths of the currents which pass through the different members of a given battery of unlike cells joined up in multiple arc. Many persons must have used the equations which I found convenient in this work, but I cannot find that any one ‡ has taken the trouble to print them all. I therefore give a few of them here, with some well known formulas introduced for the sake of clearness.

Let the internal resistances of $n$ cells, which are joined up in multiple arc with each other and with a conductor of resistance $r$, be $b_1$, $b_2$, $b_3$ . . . respectively, and the electromotive forces be $e_1$, $e_2$, $e_3$ . . .

* Stepanoff, Journal Russ. Phys. Chem. Soc., XII. 38.

† Slouginoff, Journal Russ. Phys. Chem. Soc., XIV. 2.

‡ I have not had access to the papers of Messrs. Stepanoff and Slouginoff quoted above.

Let $C$ be the current which flows through the outside resistance $r$, and let $C_k$ be the current which flows, in the same cyclic direction as $C$, through the $k$th cell. Then, if $b_k = \lambda_k\, r$,

$$C = \sum_{k=1}^{k=n} C_k; \tag{1}$$

$$b_k\, C_k + r\, C = r\,(\lambda_k\, C_k + \sum_{p=1}^{p=n} C_p) = e_k. \tag{2}$$

Let the determinant of the coefficients of the $C$'s in the set of $n$ linear equations of which (2) is an example, be denoted by $\Delta_n$; then

$$\Delta_n = r^n \begin{vmatrix} 1+\lambda_1 & 1 & 1 & 1 & \cdots & \cdot \\ 1 & 1+\lambda_2 & 1 & 1 & \cdots & \cdot \\ 1 & 1 & 1+\lambda_3 & 1 & \cdots & \cdot \\ 1 & 1 & 1 & 1+\lambda_4 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & 1+\lambda_n \end{vmatrix}$$

or

$$\Delta_n = r^n\, \delta_n.$$

It is evident that $\delta_{n+1}$ would be equal to the product of $\lambda_{n+1}$ and $\delta_n$ plus the product of all the $\lambda$'s from $\lambda_1$ to $\lambda_n$.

Let $M_n$ represent the product of all the $b$'s.

Let $N_n$ represent the sum of the $n$ products of the $b$'s taken $n - 1$ at a time.

Let $P_{n,k}$ represent the sum of the $n - 1$ products formed of all the $b$'s except $b_k$ taken $n - 2$ at a time.

Let $Q_n$ represent the sum of the $n$ products formed by multiplying every $e$ by all the $b$'s except its own.

Let $S_{n,k}$ represent the sum of the products obtained by multiplying every electromotive force except that of the $k$th cell by all the $b$'s except its own and $b_k$.

Then
$$\Delta_n = M_n + r\, N_n. \tag{3}$$

The determinant formed from the determinant of the coefficients ($\Delta_n$) of the $C$'s in the $n$ linear equations of which (2) is a type, by substituting the corresponding $e$'s for the elements of the $k$th column shall be called $\Delta_{n,k}$.

$$\Delta_{n,k} = \frac{e_k (M_n + b_k r P_{n,k}) - r b_k S_{n,k}}{b_k}. \tag{4}$$

Hence,

$$C_k = \frac{e_k (M_n + b_k r P_{n,k}) - r b_k S_{n,k}}{b_k (M_n + r N_n)}. \tag{5}$$

The current in the $k$th cell may be positive, zero, or negative according to circumstances. If the electromotive forces and the internal resistances of all the cells are given, $C_k$ will be positive if $r$ be made less than $\frac{e_k M_k}{S_{n,k} - e_k P_{n,k}}$.

The currents in all the cells will be positive if $r$ be made sufficiently small. If the electromotive forces of the cells are unequal, and if $r$ is very large, the currents in the weakest cell or cells will be negative. If the outside conductor be removed so that the poles of the battery are not connected externally, the current in the $k$th cell will be positive if $e_k$ is greater than $S_{n,k} \div P_{n,k}$. Thus, if the battery consists of three unlike cells numbered in order of descending electromotive force, $C_1$ must be positive and $C_3$ negative, but $C_2$ will be positive or not according as $e_2$ is or is not greater than $\frac{e_1 b_3 + e_3 b_1}{b_1 + b_3}$.

The case of a number of cells of the same internal resistance but of different electromotive forces joined up in multiple arc is of some interest when one has to use a number of unequally charged storage cells to send a very heavy current through an outside circuit of extremely low resistance. When the $b$'s are all equal,

$$C_k = \frac{e_k(b + n r) - r \Sigma (e)}{b (b + n r)} \quad \text{and} \quad C = \frac{\frac{\Sigma e}{n}}{\frac{b}{n} + r};$$

so that the equivalent cell has an electromotive force equal to the average of the electromotive forces of the given cells, and an internal resistance equal to $\frac{1}{n}$th of that of each cell. This case also throws some light on the properties of a thermal junction of large area formed of two plates of metal soldered together flatwise when, as is sometimes the case in practice, it is impossible to keep the whole junction at exactly the same temperature.

Equations (1) and (5) give the equation

$$C = \frac{\Sigma\left(\frac{e_k}{b_k}\right)}{1 + r\,\Sigma\left(\frac{1}{b_k}\right)} = \frac{\frac{Q}{N_n}}{\frac{M_n}{N_n} + r}, \tag{6}$$

which defines Stepanoff's equivalent cell * already mentioned. The difference of potential between the poles of the battery is

$$V = \frac{r\,Q}{M_n + r\,N_n} \tag{7}$$

When $r$ is made to increase indefinitely, $V$ approaches as a limit the electromotive force $Q / N_n$ of the equivalent cell, and $C_k$ approaches $\frac{e_k\,P_{n,k} - S_{n,k}}{N_n}$, the current in the $k$th cell when the poles of the battery are not connected by any external conductor.

If there is a battery of electromotive force $E$ in the external circuit, $r$, the quantity $E$ must, according to the direction of this external electromotive force, be added to or subtracted from the second member of each of the equations of which (2) is an example. If a battery of $n$ cells joined up in multiple arc be itself connected up parallel with a cell of electromotive force $E$ and of internal resistance $B$, $E \div B$ must be added to the numerator of the fraction which forms the second member of (6), and $r \div B$ to the denominator. Upon an examination of these cases, it appears that, if a battery of cells joined up in multiple arc be itself connected up parallel or in series with another battery, or if it be used for compensation purposes, it will exert the same influence upon the currents and differences of potential in parts of the circuit external to itself as its equivalent cell would exert.

The internal work done in the battery when its poles are connected by the external resistance, $r$, is

$$W_i = \sum_{k=1}^{k=n} (C_k^2\, b_k). \tag{8}$$

The internal work done in the equivalent cell would be

$$W_i' = \frac{M_n}{N_n}\left[\sum_{k=1}^{k=n} (C_k)\right]^2. \tag{9}$$

Let $w$ be the difference between $W_i$ and $W_i'$, and in the equation found by subtracting the members of (9) from the corresponding mem-

* See also Slouginoff, Carl's Repertorium, XVI. 539.

bers of (8), let $e_k - Cr$ be substituted for its equal $C_k\, b_k$. It will then be found that all the terms which contain $r$ disappear, since the coefficients of $Cr$ and of $C^2 r^2$ vanish identically, and that

$$w = \frac{M_n}{N_n} \sum_{k=1}^{k=n} \frac{(e_k - e_j)^2}{b_k\, b_j}; \tag{10}$$

where the sign of summation introduces once only every value of $(e_k - e_j)^2$, in which $k$ is different from $j$, and neither $k$ nor $j$ is greater than $n$.

Since $w$ does not involve $r$, it $(w)$ is the work done inside the battery when its poles are not connected by any external resistance. Equation 10 shows that $w$ cannot be negative, and that it is different from zero unless all the cells in the battery have equal electromotive forces. The expression for $w$ may be written in a form due to Slouginoff,

$$w = \sum_{k=1}^{k=n} \left( \frac{e_k^{\,2}}{b_k} \right) - \frac{e_0^{\,2}}{b_0}, \tag{11}$$

where $e_0$ and $b_0$ are the electromotive force and the internal resistance respectively of the equivalent cell.

It is known that if the poles of a battery formed of $n$ cells be connected by an outside resistance $r$, the current in the external circuit will be the same, whether the cells be joined up in series or in multiple arc, provided that

$$r = \frac{Q_n \Sigma (b_k) - M_n \Sigma (e_k)}{N_n \Sigma (e_k) - Q_n}; \tag{12}$$

or, if the cells are all alike, provided that $r = b$. It is worthy of notice, however, that the efficiencies of the battery are different in the two cases. If the cells are joined up in series, the efficiency of the battery is

$$F_s = \frac{r}{r + \Sigma (b_k)},$$

but if they are joined up in multiple arc the efficiency is

$$F_p = \frac{C^2 r}{C^2 r + \Sigma (C^2_k b_k)};$$

so that

$$\frac{F_s}{F_p} = \frac{C^2 r + \Sigma (C^2_k\, b_k)}{C^2 r + C^2 \Sigma (b_k)}. \tag{15}$$

If the cells are all alike, and if $r = b$, this ratio has the value $1 \div n$, and the arrangement in parallel is $n$ times as efficient as the arrangement in series. In the case of unlike cells, if the currents in the cells are all positive, $F_s$ is always less than $F_p$. It is easy, however, to find cases where $F_p$ is less than $F_s$, for this is true when

$$\Sigma\,(C_k^2\, b_k) < [\Sigma\,(C_k)]^2 \cdot \Sigma\,(b_k).$$

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